Stability of localized solutions in a subcritically unstable pattern-forming system under a global delayed control

B. Y. Rubinstein,[1] A. A. Nepomnyashchy,[2] and A. A. Golovin[3]
[1]*Department of Mathematics, University of California, Davis, California 95616, USA*
[2]*Department of Mathematics and Minerva Center for Nonlinear Physics of Complex Systems,Technion—Israel Institute of Technology, Haifa 32000, Israel*
[3]*Department of Engineering Sciences and Applied Mathematics, Northwestern University, Evanston, Illinois 60208-3100, USA*
(Received 28 September 2006; revised manuscript received 2 February 2007; published 25 April 2007)

The formation of spatially localized patterns in a system with subcritical instability under feedback control with delay is investigated within the framework of globally controlled Ginzburg-Landau equation. It is shown that feedback control can stabilize spatially localized solutions. With the increase of delay, these solutions undergo oscillatory instability that, for large enough control strength, results in the formation of localized oscillating pulses. With further increase of the delay the solution blows up.

DOI: 10.1103/PhysRevE.75.046213 PACS number(s): 89.75.Kd, 47.54.−r

I. INTRODUCTION

Various nonlinear extended systems are subject to saturable monotonic short-wave ("Turing") instabilities leading to the formation of stationary patterns [1–3]. The most well know examples are Rayleigh-Bénard convection and Turing patterns in reaction-diffusion systems. In the one-dimensional case, the generic equation governing the pattern envelope (amplitude) function $A(x,t)$ is the Ginzburg-Landau equation

$$\frac{\partial A}{\partial t} = \mu A + D\frac{\partial^2 A}{\partial x^2} - \kappa |A|^2 A, \quad (1)$$

with real coefficients μ, $D>0$, and $\kappa>0$. In the subcritical region, $\mu<0$, the solution $A=0$ corresponding to the equilibrium state is stable. In the supercritical region, the time evolution of the amplitude function is characterized by a monotonic decrease of a Lyapunov functional ("Ginzburg-Landau free energy") and asymptotically leads to a stable stationary spatially periodic solution with a wavenumber within the Eckhaus stability interval [4], $k^2<\mu/(3D)$. Other kinds of stationary solutions, such as spatially localized and quasiperiodic solutions, are unstable. This provides the explanation of the formation of ordered patterns from disordered initial conditions.

Some nonlinear systems exhibit nonsaturable (subcritical) instabilities, corresponding to the case $\kappa<0$ in Eq. (1). In this case, the Lyapunov functional is not bounded from below, and the solution of Eq. (1) blows up in a finite time. Of course, the description of the underlying physical problem by means of the weakly nonlinear equation (1) fails in this case. However, the blow-up of solutions can be prevented by a nonlinear feedback control. An example of such a control, which leads to the following modification of the Ginzburg-Landau equation

$$\frac{\partial A}{\partial t} = (\mu - K[A])A + D\frac{\partial^2 A}{\partial x^2} - \kappa |A|^2 A, \quad (2)$$

where $K[A]=p\,\max_x|A(x,t)|$, $p>0$, has been considered in Ref. [5], where it was applied for modeling the suppression of a morphological instability of a solidification front. It was shown that the stability properties of stationary solutions of Eq. (2) significantly differed from those of Eq. (1). All the spatially-periodic solutions of Eq. (1) turned out to be unstable, while the only stable solution observed in numerical simulations corresponded to a localized one.

Usually, in systems with feedback control, there is a delay between the measurement of the system parameters by sensors and the application of control action by actuators. In some systems this delay is small and can be neglected. The analysis described above is valid for this case. In the present paper, we consider the general case when a delay in feedback control is present. Thus, we consider a more general nonlinear control, $K[A]=p\,\max_x|A(x,t-\tau)|$, $p>0$ where τ=const is the control delay. Obviously, the stationary solutions of Eq. (2) are not affected by the delay. However, the control delay may change the stability properties of solutions and create new dynamic regimes.

The paper is organized as follows. In Sec. II, we present stationary localized solutions for Eqs. (1) and (2). In Sec. III, we perform the linear stability analysis of these solutions. We will show that the localized solutions are unstable in the absence of control, while in the presence of an undelayed control there can exist two branches of solutions, one of which is always stable and another one is unstable. We will also show that the delay of control may lead to an oscillatory instability of the localized solutions, and find the linear stability boundary $\tau(p)$. Section IV is devoted to nonlinear simulations of finite-amplitude pulse oscillations. Section V contains concluding remarks.

II. STATIONARY LOCALIZED SOLUTIONS

For $\kappa<0$, upon rescaling, one can rewrite Eq. (2) as

$$\frac{\partial A}{\partial t} = (s - K[A])A + \frac{\partial^2 A}{\partial x^2} + |A|^2 A, \quad (3)$$

where

$$s=\mathrm{sgn}(\mu),\quad K[A]=p\max_x|A(t-\tau)|,$$

without a loss of generality.

Equation (3) has a stationary localized solution

$$A(x)=A_0(x)=R(x-x_0)e^{i\Theta},\tag{4}$$

where x_0 and Θ are arbitrary constants,

$$R(y)=\sqrt{2}k(q)\mathrm{sech}[k(q)y],\quad -\infty<y<\infty,\tag{5}$$

and $k(q)$ is a positive root of the quadratic equation

$$k^2-2kq+s=0,\quad q=p/\sqrt{2}.\tag{6}$$

In the subcritical region, $s=-1$, there exists only one solution branch

$$k(q)=q+\sqrt{q^2+1},\quad -\infty<q<\infty.\tag{7}$$

Specifically, for $q=0$, i.e., in the absence of control, the localized solution has the form

$$R(y)=\sqrt{2}\ \mathrm{sech}\, y,\quad -\infty<y<\infty.\tag{8}$$

In the supercritical region, $s=1$, there are two branches of solutions

$$k(q)=q\pm\sqrt{q^2-1},\quad q\geqslant 1.\tag{9}$$

Note that for any localized solution the effective linear growth rate

$$\sigma_0=s-K[A_0]=s-2qk(q)=-[k(q)]^2<0\tag{10}$$

in the whole region of the localized solution existence.

III. STABILITY OF LOCALIZED SOLUTIONS

A. Formulation of the problem

Obviously, the stability of a localized solution does not depend on x_0 and Θ. Below, we set $x_0=\Theta=0$, and consider real $A_0(x)=R(x)$. In order to investigate the stability, we consider the evolution of a disturbance on the background of the stationary solution. Linearizing Eq. (3) around the localized solution (4),

$$A(x,t)=A_0(x)+\tilde{A}(x,t),\tag{11}$$

we find

$$\frac{\partial\tilde{A}(x,t)}{\partial t}=\frac{\partial^2\tilde{A}(x,t)}{\partial x^2}+[s-2qk(q)+2A_0^2(x)]\tilde{A}(x,t)+A_0^2(x)\tilde{A}^*(x,t)-2qk(q)\mathrm{Re}\,\tilde{A}(0,t-\tau).\tag{12}$$

It is assumed that $|\tilde{A}(x,t)|$ is bounded for $x\to\pm\infty$. Define $\tilde{A}(x,t)=\tilde{A}_r(x,t)+i\tilde{A}_i(x,t)$, where $\tilde{A}_r$ and $\tilde{A}_i$ are real functions. The problems for $\tilde{A}_r$ and $\tilde{A}_i$ are decoupled:

$$\frac{\partial\tilde{A}_r(x,t)}{\partial t}=\frac{\partial^2\tilde{A}_r(x,t)}{\partial x^2}+[s-2qk(q)+3A_0^2(x)]\tilde{A}_r(x,t)-2qk(q)\tilde{A}_r(0,t-\tau),\tag{13}$$

$$\frac{\partial\tilde{A}_i(x,t)}{\partial t}=\frac{\partial^2\tilde{A}_i(x,t)}{\partial x^2}+[s-2qk(q)+A_0^2(x)]\tilde{A}_i(x,t).\tag{14}$$

Here,

$$A_0(x)=\sqrt{2}k(q)\mathrm{sech}[k(q)x].$$

Introduce a new coordinate $z\equiv k(q)x$ and consider normal modes

$$\tilde{A}_r(x,t)=u(z)e^{\sigma t},\quad \tilde{A}_i(x,t)=v(z)e^{\sigma t}.$$

Taking into account relation (6), one obtains an equation which is valid in both subcritical and supercritical cases:

$$k^2u''+\left(-k^2-\sigma+\frac{6k^2}{\cosh^2 z}\right)u=2kq\frac{u(0)e^{-\sigma\tau}}{\cosh z};\quad |u<\infty,\ z\to\pm\infty,\tag{15}$$

$$k^2v''+\left(-k^2-\sigma+\frac{2k^2}{\cosh^2 z}\right)v=0;\quad |v|<\infty,\ z\to\pm\infty,\tag{16}$$

where a prime means the differentiation with respect to z. The problem (15) describes amplitude disturbances of the localized solution, while Eq. (16) describes its phase disturbances.

B. Phase disturbances

Let us start with the problem (16). Rewrite it as

$$-v''+(1-2\cosh^{-2}z)v=-(\sigma/k^2)v,\quad |v|<\infty,\ z\to\pm\infty,\tag{17}$$

to obtain the well-known eigenvalue problem for the Schrödinger equation, which is exactly solvable (see, e.g., Ref. [6] or [7]). The continuum spectrum of the problem is $-(\sigma/k^2)>1$, hence, it does not produce any instability. The only discrete eigenvalue is $\sigma=0$, with the eigenfunction

$$v(z)=\mathrm{sech}\, z,$$

which corresponds to an infinitesimal change of Θ in Eq. (4).

C. Amplitude disturbances

In the present subsection, we analyze the nonlocal eigenvalue problem (15).

1. Stability in the absence of the control

In the case $q=0$ (no control), the localized solution exists only in the subcritical region, $s=-1$, and is described by Eqs. (4) and (5) with $k=1$. The eigenvalue problem (15) can be written as

$$-u''+(1-6\,\mathrm{sech}^2 z)u=-\sigma u;\quad |u|<\infty,\quad z\to\pm\infty.\tag{18}$$

Again, the continuum spectrum of the problem is located at $\sigma<-1$ and does not produce any instability. The discrete spectrum includes two eigenvalues [6,7]

$$\sigma = 0,\ u = \sinh z\,\mathrm{sech}^2 z,$$

$$\sigma = 3, \quad u = \mathrm{sech}^2 z.$$

The first mode corresponds to a translation of the localized solution [an infinitesimal change of x_0 in Eq. (4)]. The second mode results in the instability of the subcritical localized solution in the absence of control.

2. Stability in the presence of the control

Now consider Eq. (15) in the case when control is present, $q\neq 0$. Obviously, $\sigma < -k^2 < 0$ for any disturbances which do not decay as $z\to\infty$. Hence, for the stability analysis it is sufficient to consider localized solutions with $\mathrm{Re}\,\sigma > -k^2$. Due to the symmetry of Eq. (15), any eigenfunctions can be represented by either even or odd functions of z. For odd eigenfunctions $u(0)=0$ and one returns to the uncontrolled case discussed above. Therefore, later on we consider only even solutions of Eq. (15).

a. Analytical solution of the linear stability problem. One can fix the norm of the eigenfunction $u(z)$ by the condition $u(0)=1$, and present the eigenvalue problem in the form

$$u'' + \left(\frac{6}{\cosh^2 z} - r^2\right)u = m\frac{\exp[(1-r^2)\tau/(m-1)]}{\cosh z},$$
$$-\infty < z < \infty, \tag{19}$$

$$|u| \to 0, \quad z \to \pm\infty, \tag{20}$$

where

$$r^2 = \frac{\sigma + k^2}{k^2}, \quad m = \frac{s+k^2}{k^2}.$$

According to Eq. (7), in the subcritical region $s=-1$,

$$m = \frac{2q}{q+\sqrt{q^2+1}}.$$

Hence, $0 < m < 1$ for the stabilizing control ($q>0$), and $-\infty < m < 0$ for the destabilizing control ($q<0$). In the supercritical region $s=1$,

$$m = \frac{2q}{q\pm\sqrt{q^2-1}},$$

where $q \geq 1$ [see Eq. (9)]. One can see that $1 < m \leq 2$ for the upper branch and $2 \leq m < \infty$ for the lower branch. Thus, the stability of localized solutions in all the cases mentioned in Sec. II can be studied using of Eq. (19).

The eigenvalue σ is above the continuous spectrum if $\mathrm{Re}(r^2) > 0$. The instability corresponds to $\mathrm{Re}(r^2) > 1$.

The general solution of Eq. (19) can be written as

$$u(z) = u_0(z) + u_p(z),$$

where u_0 is the general solution of the homogeneous equation and u_p is a particular solution of the inhomogeneous equation. Since we are interested only in even solutions of the problem, it is sufficient to consider the region $0 \leq z < \infty$.

The general solution of the homogeneous equation is

$$u_0(z) = C_+^h P_2^r(\tanh z) + C_-^h P_2^{-r}(\tanh z), \tag{21}$$

where $P_n^m(x)$ denotes the associated Legendre polynomial. The particular solution of the inhomogeneous equation can be found using variation of parameters, which gives

$$u_0(z) = C_+^i(z) P_2^r(\tanh z) + C_-^i(z) P_2^{-r}(\tanh z), \tag{22}$$

where

$$C_\pm^i(z) = \pm\frac{\pi m \exp[(1-r^2)\tau/(m-1)]}{2\sin\pi r}\int\frac{P_2^{\mp r}(\tanh z)}{\cosh z}dz$$
$$= \pm\frac{\pi m \exp[(1-r^2)\tau/(m-1)]}{2\sin\pi r}\int\frac{P_2^{\mp r}(y)}{\sqrt{1-y^2}}dy. \tag{23}$$

The computation of the integral in Eq. (23) gives

$$C_\pm^i = \pm\frac{\pi m\exp[(1-r^2)\tau]/(m-1)}{2\Gamma(3\pm r)\sin\pi r}\{3[I_3^{(\mp r+3)/2}(w)$$
$$-2I_3^{(\mp r+1)/2}(w) + I_3^{(\mp r-1)/2}(w)] \pm 3p[I_2^{(\mp r+1)/2}(w)$$
$$-I_2^{(\mp r-1)/2}(w)] + (r^2-1)I_1^{(\mp r-1)/2}(w)\}, \tag{24}$$

where $w=(1+y)/(1-y)=(1+\tanh z)/(1-\tanh z)$, and

$$I_n^\alpha(w) = \int\frac{w^\alpha}{(w+1)^n}dw = \frac{w^{1+\alpha}}{1+\alpha}\,{}_2F_1(n,1+\alpha;2+\alpha;-w). \tag{25}$$

Here ${}_2F_1$ is a confluent hypergeometric function. Thus, the general solution of the problem reads

$$u(z) = (C_+^h + C_+^i)P_2^r(\tanh z) + (C_-^h + C_-^i)P_2^{-r}(\tanh z). \tag{26}$$

Condition (20) leads to following values of the coefficients:

$$C_+^h = -\frac{\pi m\exp[(1-r^2)\tau/(m-1)]}{2\Gamma(3+r)\sin\pi r}\frac{\pi(1-r^2)}{2}\sec\frac{\pi r}{2},$$

$$C_-^h = 0. \tag{27}$$

Finally, applying the condition $u(0)=1$ to Eqs. (24), (26), and (27), and using the properties of the Γ function and hypergeometric functions [8], one arrives at the following relation:

$$\frac{m\exp[(1-r^2)\tau/(m-1)]}{2r(4-r^2)}\left\{3r + \frac{r^2-1}{2}\left[\psi\left(\frac{r+1}{4}\right)\right.\right.$$
$$\left.\left.-\psi\left(\frac{r+3}{4}\right)\right]\right\} = 1, \tag{28}$$

where $\psi(x)=\Gamma'(x)/\Gamma(x)$ is the logarithmic derivative of the Γ function. Equation (28) describes the dependence of the

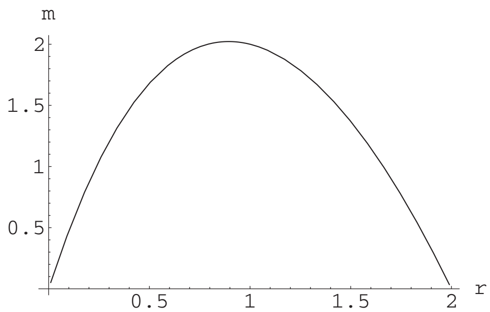

FIG. 1. Dependence $m(r)$ defined by Eq. (29).

growth rate on the control parameter for both subcritical and supercritical regions in an implicit form.

b. Stability of localized solutions under control without delay. First, let us consider the case of undelayed control, $\tau=0$. In this case,

$$\frac{m}{2r(4-r^2)}\left\{3r+\frac{r^2-1}{2}\left[\psi\left(\frac{r+1}{4}\right)-\psi\left(\frac{r+3}{4}\right)\right]\right\}=1. \tag{29}$$

For monotonic disturbances (real r), the dependence $m(r)$ can be found explicitly; it is shown in Figs. 1 and 2. One can see that for any $m<2$ there are two values of r, one of which is larger than 1. This means that the localized solutions in the subcritical region, as well as the upper branch of the localized solutions in the supercritical region are unstable.

In the region $2<m<m_*=2.02193$, both values of r are real and less than 1. For $m>m_*$ the two eigenvalues r are complex conjugate, with $\mathrm{Re}(r^2)<1$ (see Fig. 2). For large m, the leading order terms of the asymptotic expansion for $r(m)$ can be written in the form

$$r=\frac{\pi(m-4)}{4}e^{-\pi\sqrt{m-5}/2}+i\sqrt{m-5}. \tag{30}$$

Hence, the stability condition $\mathrm{Re}(r^2)<1$ is not violated in this case.

Therefore, the lower branch of the localized solutions in the supercritical region is stable in the whole region of its

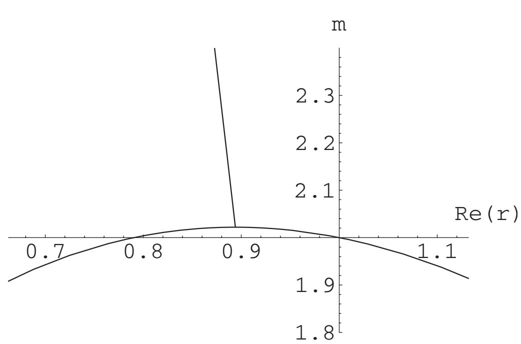

FIG. 2. Real parts of $r(m)$ defined by Eq. (29) for $m<m_*$ (two real eigenvalues) and $m>m_*$ (two complex conjugate eigenvalues).

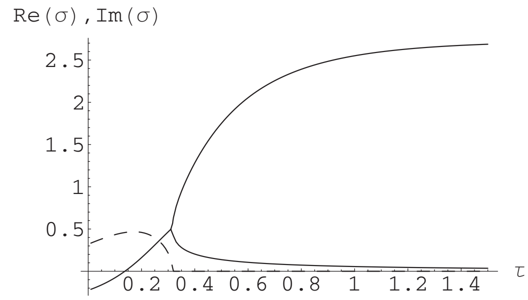

FIG. 3. Typical dispersion curves defined by Eq. (28), showing the real part (solid lines) and imaginary part (dashed line) of the perturbation growth rate σ as functions of the control delay τ.

existence, $m>2$, i.e., for $p>\sqrt{2}$. The stability of the localized solutions under the global control was recently observed in numerical simulations [5]. Note that the neutral disturbance ($r=1$) corresponding to the merging point of the two branches ($m=2$) can be expressed by means of elementary functions, $u(z)=\mathrm{sech}\,z-z\sinh z\,\mathrm{sech}^2 z$.

c. Stability of localized solutions under delayed control. Let us consider now the case of delayed control, $\tau\neq 0$, described by Eq. (28). Obviously, the monotonic stability boundary $r^2=1$ is not changed by the delay. Hence, the boundary between monotonically stable and monotonically unstable solutions $m=2$ is unchanged. Therefore, the localized solutions at the upper branch are unstable. However, the delay can produce an oscillatory instability of the supercritical localized solution corresponding to the lower branch. A typical dependence of the growth rate σ on the delay parameter τ for a fixed value of p is shown in Fig. 3. The growth rate of a monotonic mode cannot cross the value $\sigma=0$ at a finite value of τ but the real part of the growth rate of the oscillatory mode does cross zero. Note that for sufficiently large τ the pair of complex conjugate eigenvalues with $\mathrm{Re}(\sigma)>0$ is transformed into a pair of real positive eigenvalues, i.e., the instability of the stationary localized solution becomes monotonic.

The region of stability of the localized solution is shown in Fig. 4 bounded by the solid line. The end point of the oscillatory instability boundary (marked by a star) corresponds to $p=\sqrt{2}$ and $\tau=(1-\ln 2)/3\approx 0.102$. For $p\to\infty$, $\tau\to\pi/2$.

IV. NUMERICAL SIMULATIONS

We have performed numerical simulations of Eq. (3) with delayed control term, by means of a pseudospectral code with periodic boundary conditions and time integration in Fourier space, using Crank-Nicholson scheme for the linear operator and Adams-Bashforth one for the nonlinear one. Figure 5(a) shows the spatiotemporal diagram of the solution for the parameter values inside the stability domain (see Fig. 4). One observes the formation of a stationary localized solution after transient oscillations. With the increase of the delay above the critical value $\tau_c(p)$ (solid line in Fig. 4), the localized solution becomes unstable with respect to oscilla-

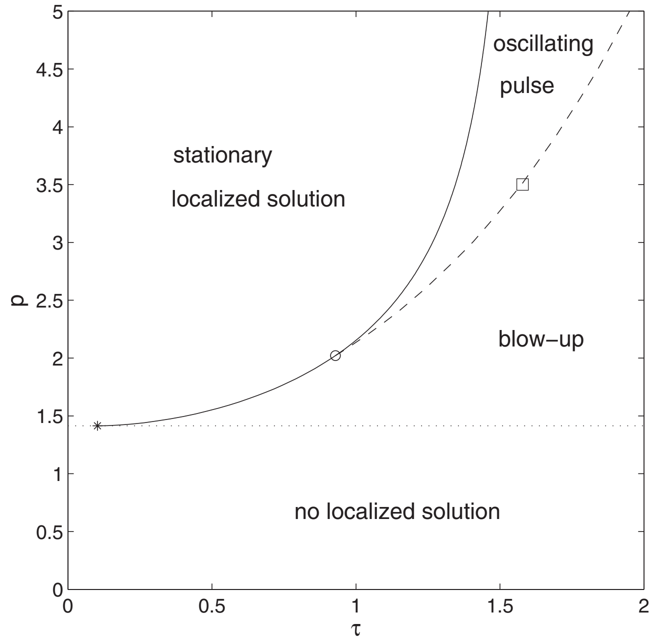

FIG. 4. Stability boundary of the localized solution (4)–(6) for $s=1$ (solid line), and the blow-up boundary of the oscillating solution resulting from the oscillatory instability of the solution (4)–(6) (dashed line). The star corresponds to $p=\sqrt{2}$ and $\tau=(1-\ln 2)/3$. The circle corresponds to the merge point of the stability and the blow-up boundary found numerically $p\approx 2.022$, $\tau\approx 0.929$. The square at $p\approx 3.5$, $\tau\approx 1.575$ corresponds to the formation of a homoclinic cycle. Dotted line corresponds to $p=\sqrt{2}$ below which no localized solutions exist.

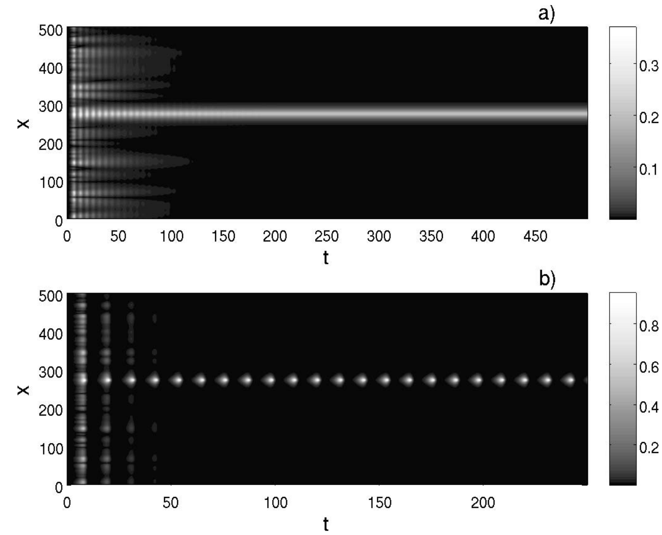

FIG. 5. Spatiotemporal diagrams of numerical solutions of Eq. (3) for $p=5.0$ and (a) $\tau=1.4$ (stationary localized solution), (b) $\tau=1.9$ (oscillating localized pulse). The amplitude $|A|$ is shown.

tory instability leading to the formation of an oscillating localized pulse. The formation of such pulse is shown in Fig. 5(b).

Figure 6(a) shows examples of phase portraits of the oscillations at the pulse maximum for different values of the delay for a fixed value of the control parameter, $p=5.0$. One can see that when the delay passes through the critical value τ_c, the oscillation amplitude increases as $(\tau-\tau_c)^{1/2}$ [see Fig. 6(b)] and then further grows with the increase of τ until the solution blows up for $\tau>\tau_b$. A similar scenario occurs for smaller values of the control parameter p. We have found that stable oscillating pulses are formed for $p>p_*\approx 2.022$ and $\tau_c(p)<\tau<\tau_b(p)$. The boundary $\tau_b(p)$ is shown in Fig. 4 by the dashed line. We have also found that for $p<p_*$ the solution either tends to a stationary localized solution (4)–(6) for $\tau<\tau_c(p)$ or blows up for $\tau>\tau_c(p)$.

Figure 7 shows the oscillations of the pulse maximum for two values of the control parameter p, and delay τ right near the blow-up boundary τ_b. One can see that for smaller values of p the oscillations look like a limit cycle whereas for larger values of p, near the blow-up point, the oscillations resemble a homoclinic loop originating from zero. Indeed, in this case, as one can see from Fig. 7, the oscillation period increases and the system spends large time near the point $|A|=0$. Apparently, blow up is caused by the disappearance of the stable limit cycle solution. We have found that the oscillation amplitude near the blow-up boundary decreases sharply with p approaching the critical point p_*, and the minimal amplitude approaches zero (becomes less than 0.005) for $p>p_h\approx 3.5$.

Thus, we suggest the following bifurcation scenario. For $p<p_*\approx 2.022$ (shown in Fig. 4 by a circle) the stationary localized solution exhibits subcritical Hopf bifurcation at $\tau=\tau_c(p)$ and the solution blows up at $\tau>\tau_c(p)$. For $p>p_*$ the stationary localized solution exhibits a supercritical Hopf bifurcation at $\tau=\tau_c(p)$ which leads to the formation of an oscillating localized pulse for $\tau>\tau_c(p)$ that corresponds to a

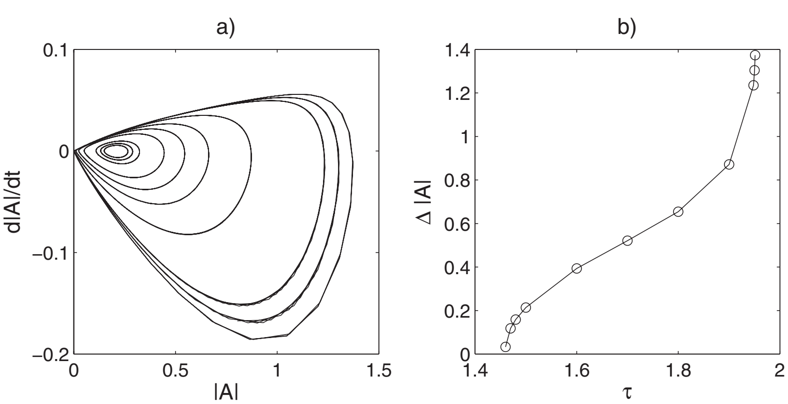

FIG. 6. Results of numerical simulations of Eq. (3) showing: (a) phase portrait of oscillating localized pulses for $p=5.0$ and $\tau=1.47, 1.48, 1.5, 1.6, 1.7, 1.8, 1.9, 1.948, 1.950, 1.951$; the values of $|A|$ correspond to the spatial location of the pulse maximum. (b) amplitude of an oscillating localized pulse, $\Delta|A|$, as a function of delay τ.

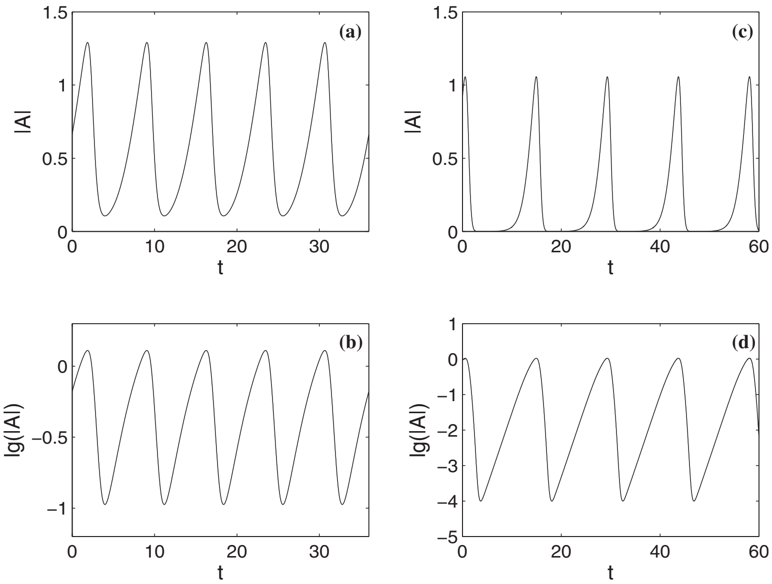

FIG. 7. Oscillations of the localized pulse maximum right near the blow-up boundary for $p=2.2$, $\tau=1.0382$ (a), (b) and $p=5.0$, $\tau=1.949$ (c), (d); for $p=2.2$, $\tau_b=1.0384$ and for $p=5.0$, $\tau_b=1.9514$.

stable limit cycle. With further increase of τ, for $\tau > \tau_b(p)$, this solution disappears leading to a blow-up by two possible scenarios, depending on the control parameter p. For $p_* < p < p_h \approx 3.5$ (shown by a square in Fig. 4) the oscillating localized solution disappears as a result of the saddle-node bifurcation when the stable and unstable limit cycles merge. For $p > p_h$ and $\tau > \tau_b$ the solution blows up as a result of a homoclinic bifurcation.

V. CONCLUSIONS

We have investigated the dynamics of subcritically unstable pattern forming systems under the action of a delayed global feedback control within the framework of the controlled Ginzburg-Landau equation for the pattern envelope function (3). The control is based on the measurement of the pattern maximum amplitude. We have shown that the feedback control stabilizes a stationary localized solution of Eq. (3) determined by Eqs. (4)–(6) that leads to formation of spatially localized patterns. The localized solution can exist only if the value of the control parameter is larger than $\sqrt{2}$. We have performed a linear stability analysis of this solution and obtained an analytic dispersion relation (28) that determines the perturbation growth rate on the values of the control strength and delay. The linear stability analysis shows that with the increase of delay, for $\tau > \tau_c(p)$, the stationary localized solution exhibits an oscillatory instability leading to the formation of spatially localized oscillating pulses corresponding to localized oscillating patterns. The stability boundary $\tau_c(p)$ is found analytically. We have performed numerical simulations of Eq. (3) that confirmed these conclusions. By means of numerical simulations we have found that the formation of delay-driven localized oscillating pulses is possible for $p > p_* \approx 2.022$ by a supercritical Hopf bifurcation. For $p < p_*$ the bifurcation is subcritical and the solution blows up for $\tau > \tau_c$. We have also shown that for $p > p_*$ the oscillatory localized pulses are stable for $\tau_c(p) < \tau < \tau_b(p)$. For $\tau > \tau_b(p)$ the solution blows up and the blow-up boundary $\tau_b(p)$ is found numerically. We have observed that for $p < p_h \approx 3.5$ the blow-up results from the merge of stable and unstable limit cycles (saddle-node bifurcation) and for $p > p_h$ it results from the transformation of a stable limit cycle into a homoclinic loop.

ACKNOWLEDGMENTS

A.A.N. acknowledges the support by B. and G. Greenberg Research Fund (Ottawa) and the support of the Israel Science Foundation Grant No. 812/06. A.A.G. acknowledges the support of the US National Science Foundation, Grant No. DMS-0505878.

[1] M. C. Cross and P. C. Hohenberg, Rev. Mod. Phys. **65**, 851 (1993).
[2] D. Walgraef, *Spatio-Temporal Pattern Formation with Examples from Physics, Chemistry and Materials Science* (Springer-Verlag, Berlin, 1997).
[3] C. Bowman and A. C. Newell, Rev. Mod. Phys. **70**, 289 (1998).
[4] W. Eckhaus, *Studies in Non-linear Stability Theory* (Springer-Verlag, Berlin, 1965).
[5] A. A. Nepomnyashchy, A. A. Golovin, V. Gubareva, and V. Panfilov, Physica D **199**, 61 (2004).
[6] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics: Nonrelativistic Theory* (Pergamon, Oxford, 1965).
[7] A. D. Polyanin and V. F. Zaitsev, *Handbook of Exact Solutions for Ordinary Differential Equations* (Chapman & Hall/CRC, Boca Raton, 2003).
[8] A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev, *Integral and Series, Vol. 3: More Special Functions* (Gordon and Breach Science Publishers, New York, 1990).